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LETTER TO THE EDITOR

$SO(6,2)$ dynamical symmetry of the $SU(2)$ MIC-Kepler problem

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Abstract. It is shown that the full group of dynamical symmetry for the 5D $SU(2)$ MIC-Kepler problem is $SO(6,2)$.

It is well known that both nonrelativistic and relativistic quantum Kepler problems (with or without magnetic charges) can be treated in the terms of the dynamical group $SO(4,2)$ [1, 2]. The dynamical symmetry properties of the Kepler and MIC-Kepler [3] problems have been considered in detail in [4].

Non-Abelian generalization of the nonrelativistic MIC-Kepler and Kepler-monopole problems is possible in certain higher dimensions [5]. So, the $SU(2)$ generalization is available in 5D Euclidean space. It is known [6–8] that the 5D MIC-Kepler (Kepler) model on the background of the $SU(2)$ Yang–Mills instantonic potential can be formulated in terms of an 8D harmonic (singular) oscillator. Further consideration of this correspondence is motivated because the complicated dynamics in such a topologically nontrivial background as the Yang–Mills instanton (which is of great interest in physics [9]) can be treated in merely algebraic terms. In [6, 10] it has been shown that $SU(2)$ Kepler problem manifests $SO(6)$ symmetry. On the other hand, it possesses the $SO(1,2)$ ($SU(1,1)$) dynamical symmetry as does its MIC-Kepler counterpart [8].

In this letter we demonstrate that the full dynamical group of the 5D $SU(2)$ MIC-Kepler problem is $SO(6,2)$ (the Kepler problem does not possess such a larger symmetry). We formulate this symmetry in terms of the 8D harmonic oscillator creation and annihilation operators and show how to derive the known $SO(6)$ symmetry (it is not the subgroup!) using such notions.

We recall that the 8D harmonic (singular) oscillator eigenproblem is described as

$$H_0\Psi_0^{(8)} = E_0\Psi_0^{(8)} \qquad H_0 = -2\frac{\partial^2}{\partial\xi_i\,\partial\xi_i^*} + \frac{\omega^2}{2}\xi_i^*\xi_i \tag{1}$$

$$H\Psi^{(8)} = E\Psi^{(8)} \qquad H = H_0 - \frac{2\boldsymbol{K}^2}{\xi_i^*\xi_i} \qquad i = 1, 2, 3, 4 \tag{2}$$

with

$$\boldsymbol{K}^2 = K_\alpha K_\alpha \qquad \alpha = 1, 2, 3 \tag{3}$$

where K_a in terms of the auxiliary coordinates

$$z_1 = -\frac{\mathrm{Im}\,\xi_2}{\mathrm{Re}\,\xi_1} \qquad z_2 = \frac{\mathrm{Re}\,\xi_2}{\mathrm{Re}\,\xi_1} \qquad z_3 = -\frac{\mathrm{Im}\,\xi_1}{\mathrm{Re}\,\xi_1} \tag{4}$$

is expressed as

$$K_\alpha = \frac{\mathrm{i}}{2}\left(z_\alpha z_\beta \frac{\partial}{\partial z_\beta} + \frac{\partial}{\partial z_\alpha} - \varepsilon_{\alpha\beta\gamma} z_\beta \frac{\partial}{\partial z_\gamma}\right). \tag{5}$$

Under certain conditions it is equivalent to the 5D $SU(2)$ MIC-Kepler (Kepler) problem

$$\mathcal{H}_0\varphi_0^{(5)} = \mathcal{E}_0\varphi_0^{(5)} \qquad \mathcal{H}_0 = \frac{\pi_\mu^2}{2} + \frac{l(l+1)}{2R^2} - \frac{\kappa}{R} \tag{6}$$

$$\mathcal{H}\varphi^{(5)} = \mathcal{E}\varphi^{(5)} \qquad \mathcal{H} = \mathcal{H}_0 - \frac{l(l+1)}{2R^2} \tag{7}$$

where the covariant derivative $\pi_\mu = -\mathrm{i}\partial_\mu - A_\mu^a \Lambda_a^{2l+1}$ contains $SU(2)$ Yang–Mills instanton [9] as the gauge potential defined due to

$$A_\mu^a dr_\mu = \frac{1}{R(R+r_0)}(-r_4 dr_a + r_a dr_4 - \varepsilon_{abc} r_b dr_c)$$

and Λ_a^{2l+1} are the generators of the $(2l+1)$ -dimensional representation of $SU(2)$.

These conditions are:

(1) the coordinates of 5D Euclidean space are expressed through those of 8D space by means of the Hurwitz transformation

$$r_\mu = \xi^* \gamma_\mu \xi$$
$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_\alpha = \begin{pmatrix} 0 & -\mathrm{i}\sigma_\alpha \\ i\sigma_\alpha & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

which possesses the property $R = \xi^*\xi$;

(2) the eigenvalues of one problem are expressed through the parameters of another one and vice versa:

$$E_0 = 4\kappa \qquad \omega^2 = -8\mathcal{E}_0 \tag{9}$$

$$E = 4\kappa \qquad \omega^2 = -8\mathcal{E} \tag{10}$$

(3) the equivariance condition

$$\boldsymbol{K}^2\Psi^{(8)} = l(l+1)\Psi^{(8)} \tag{11}$$

is supposed to hold. It allows us to establish the correspondence between the respective Hilbert spaces

$$\Psi^{(8)}(\xi_i, \xi_i^*) = \sum_{m,m'} D_{mm'}^l(\boldsymbol{z})\varphi_{mm'}^{(5)}(r_\mu). \tag{12}$$

Here $D_{mm'}^l(\boldsymbol{z})$ are the $SU(2)$ Wigner functions expressed through the vector parameters which are related to the Euler angles as

$$\begin{aligned} z_1 &= \tan\frac{\theta}{2}\cos\frac{\varphi-\psi}{2}\Big/\cos\frac{\varphi+\psi}{2} \\ z_2 &= \tan\frac{\theta}{2}\sin\frac{\varphi-\psi}{2}\Big/\cos\frac{\varphi+\psi}{2} \\ z_3 &= \tan\frac{\varphi+\psi}{2}. \end{aligned} \tag{13}$$

The 8D oscillator's annihilation and creation operators

$$a_i = \sqrt{\frac{\omega}{2}}\left(\xi_i + \frac{1}{\omega}\frac{\partial}{\partial \xi_i^*}\right) \qquad b_i = \sqrt{\frac{\omega}{2}}\left(\xi_i^* + \frac{1}{\omega}\frac{\partial}{\partial \xi_i}\right) \tag{14}$$

$$a_i^\dagger = \sqrt{\frac{\omega}{2}}\left(\xi_i^* - \frac{1}{\omega}\frac{\partial}{\partial \xi_i}\right) \qquad b_i^\dagger = \sqrt{\frac{\omega}{2}}\left(\xi_i - \frac{1}{\omega}\frac{\partial}{\partial \xi_i^*}\right) \tag{15}$$

satisfy the standard relations

$$[a_i, a_j^\dagger] = [b_i, b_j^\dagger] = \delta_{ij}. \tag{16}$$

Their quadratic combinations

$$R_{ij} = a_i^\dagger a_j + b_i b_j^\dagger \qquad L_{ij} = a_i^\dagger a_j - b_i b_j^\dagger \tag{17}$$

$$N_{ij} = b_i a_j + a_i^\dagger b_j^\dagger \qquad Q_{ij} = \mathrm{i}(b_i a_j - a_i^\dagger b_j^\dagger) \tag{18}$$

constitute the algebra which is isomorphic to $u(4,4)$

$$\begin{aligned}
&[L_{ij}, R_{kn}] = \delta_{kj} R_{in} - \delta_{in} R_{kj} \qquad [L_{ij}, Q_{kn}] = \delta_{kj} Q_{in} - \delta_{in} Q_{kj} \\
&[L_{ij}, N_{kn}] = \delta_{kj} N_{in} - \delta_{in} N_{kj} \qquad [R_{ij}, Q_{kn}] = -\mathrm{i}(\delta_{kj} N_{in} + \delta_{in} N_{kj}) \\
&[R_{ij}, N_{kn}] = \mathrm{i}(\delta_{kj} Q_{in} + \delta_{in} Q_{kj}) \qquad [Q_{ij}, N_{kn}] = \mathrm{i}(\delta_{kj} R_{in} + \delta_{in} R_{kj}) \\
&[L_{ij}, L_{kn}] = \delta_{kj} L_{in} - \delta_{in} L_{kj} \qquad [R_{ij}, R_{kn}] = \delta_{kj} L_{in} - \delta_{in} L_{kj} \\
&[Q_{ij}, Q_{kn}] = -\delta_{kj} L_{in} + \delta_{in} L_{kj} \qquad [N_{ij}, N_{kn}] = -\delta_{kj} L_{in} + \delta_{in} L_{kj}.
\end{aligned}$$

We introduce the operators

$$\begin{aligned}
&\mathcal{L}_{\mu\nu} = \tfrac{1}{2}\Omega_{\mu\nu}^{ij} L_{ij} = \mathcal{J}_{\mu\nu} = \mathrm{i}[R\pi_\mu, R\pi_\nu] \\
&\mathcal{L}_{\mu 0} = -\tfrac{1}{2}\gamma_\mu^{ij} N_{ij} = \mathcal{A}_\mu = \tfrac{1}{2}(\mathcal{Y}_\mu - r_\mu) \\
&\mathcal{L}_{\mu 6} = \tfrac{1}{2}\gamma_\mu^{ij} R_{ij} = \mathcal{M}_\mu = \tfrac{1}{2}(\mathcal{Y}_\mu + r_\mu) \\
&\mathcal{L}_{\mu 7} = -\tfrac{1}{2}\gamma_\mu^{ij} Q_{ij} = \Gamma_\mu = R\pi_\mu \\
&\mathcal{L}_{70} = \frac{1}{2}\delta^{ij} R_{ij} = \Gamma_0 = \frac{1}{2}\left(R\pi^2 + \frac{\boldsymbol{K}^2}{R} + R\right) \\
&\mathcal{L}_{76} = -\frac{1}{2}\delta^{ij} N_{ij} = \Gamma_6 = \frac{1}{2}\left(R\pi^2 + \frac{\boldsymbol{K}^2}{R} - R\right) \\
&\mathcal{L}_{60} = -\tfrac{1}{2}\delta^{ij} Q_{ij} = \mathcal{T} = r_\mu \pi_\mu - 2\mathrm{i}
\end{aligned} \tag{19}$$

with

$$\mathcal{Y}_\mu = r_\mu\left(\pi^2 + \frac{\boldsymbol{K}^2}{R^2}\right) + \{\pi_\lambda, \mathcal{J}_{\lambda\mu}\} \tag{20}$$

where γ_μ^{ij} are the components of the matrices γ_μ defined in (8) and $\Omega_{\mu\nu}^{ij} = (\frac{1}{2\mathrm{i}}[\gamma_\mu, \gamma_\nu])^{ij}$. The brackets $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ denote the commutator and anticommutator, respectively. (When written in 5D terms, the operators (19) contain the auxiliary coordinates (4) within $\boldsymbol{K}^2$, i.e. the equivariance condition has not been applied yet! But it does not matter, because $\boldsymbol{K}^2$ commutes with all of $\mathcal{L}_{ab}$.)

The operators (19) satisfy $SO(6,2)$ commutation relations

$$\begin{aligned}
&[\mathcal{L}_{ab}, \mathcal{L}_{cd}] = \mathrm{i}(g_{bc}\mathcal{L}_{da} - g_{da}\mathcal{L}_{bc} + g_{db}\mathcal{L}_{ac} - g_{ac}\mathcal{L}_{db}) \\
&(g_{ab}) = \mathrm{diag}(-1, 1, 1, 1, 1, 1, 1, -1) \\
&a, b, c, d = 0, \ldots, 7.
\end{aligned} \tag{21}$$

We assert that $SO(6,2)$ is the group of dynamical symmetry of the 5D $SU(2)$ Kepler problem because it contains the subgroup $SO(1,2)$ generated by $(\Gamma_0, \Gamma_6, \mathcal{T})$

$$[\Gamma_0, \Gamma_6] = \mathrm{i}\mathcal{T} \qquad [\Gamma_6, \mathcal{T}] = -\mathrm{i}\Gamma_0 \qquad [\mathcal{T}, \Gamma_0] = \mathrm{i}\Gamma_6 \tag{22}$$

which is isomorphic to the group generated by

$$\begin{aligned}
H_0 &= 2(\Gamma_0 + \Gamma_6) + \frac{\omega^2}{2}(\Gamma_0 - \Gamma_6) \\
B_{20}^\dagger &= -\mathrm{i}\left(\frac{\omega^2}{2}(\Gamma_0 - \Gamma_6) - 2(\Gamma_0 + \Gamma_6) - 2\mathrm{i}\omega\mathcal{T}\right) \\
B_{20} &= \mathrm{i}\left(\frac{\omega^2}{2}(\Gamma_0 - \Gamma_6) - 2(\Gamma_0 + \Gamma_6) + 2\mathrm{i}\omega\mathcal{T}\right)
\end{aligned} \tag{23}$$

with $SO(1,2)$ commutation relations

$$\left[\frac{H_0}{2\omega}, \frac{B_{20}^\dagger}{2\omega}\right] = \mathrm{i}\frac{B_{20}^\dagger}{2\omega} \qquad \left[\frac{H_0}{2\omega}, \frac{B_{20}}{2\omega}\right] = -\mathrm{i}\frac{B_{20}}{2\omega} \qquad \left[\frac{B_{20}}{2\omega}, \frac{B_{20}^\dagger}{2\omega}\right] = 2\frac{H_0}{2\omega}. \tag{24}$$

The algebra (23) generates the spectrum of the 8D harmonic oscillator

$$E_{0N} = \omega(N+4) \qquad N = 0, 1, 2, \ldots \tag{25}$$

and due to the relation (10) one can obtain the spectrum of the 5D $SU(2)$ MIC-Kepler problem

$$\mathcal{E}_{0N} = -\frac{\kappa^2}{2(\frac{N}{2}+2)^2}. \tag{26}$$

In the case of the 5D $SU(2)$ Kepler problem [8] the generators

$$\Gamma_0' = \tfrac{1}{2}(R\pi^2 + R) \qquad \Gamma_6' = \tfrac{1}{2}(R\pi^2 - R) \qquad \mathcal{T}' = r_\mu\pi_\mu - 2\mathrm{i} \tag{27}$$

can be introduced. They satisfy the $SO(1,2)$ commutation relations

$$[\Gamma_0', \Gamma_6'] = \mathrm{i}\mathcal{T}' \qquad [\Gamma_6', \mathcal{T}'] = -\mathrm{i}\Gamma_0' \qquad [\mathcal{T}', \Gamma_0'] = \mathrm{i}\Gamma_6' \tag{28}$$

and are related to the generators H, $B_2^\dagger \equiv B_{20}^\dagger - \mathrm{i}\frac{2l(l+1)}{\xi^*\xi}$, $B_2 \equiv B_{20} + \mathrm{i}\frac{2l(l+1)}{\xi^*\xi}$, satisfying

$$\left[\frac{H}{2\omega}, \frac{B_2^\dagger}{2\omega}\right] = \mathrm{i}\frac{B_2^\dagger}{2\omega} \qquad \left[\frac{H}{2\omega}, \frac{B_2}{2\omega}\right] = -\mathrm{i}\frac{B_2}{2\omega} \qquad \left[\frac{B_2}{2\omega}, \frac{B_2^\dagger}{2\omega}\right] = 2\frac{H}{2\omega} \tag{29}$$

as

$$\begin{aligned}
H &= 2(\Gamma_0' + \Gamma_6') + \frac{\omega^2}{2}(\Gamma_0' - \Gamma_6') \\
B_2^\dagger &= -\mathrm{i}\left(\frac{\omega^2}{2}(\Gamma_0' - \Gamma_6') - 2(\Gamma_0' + \Gamma_6') - 2\mathrm{i}\omega\mathcal{T}'\right) \\
B_2 &= \mathrm{i}\left(\frac{\omega^2}{2}(\Gamma_0' - \Gamma_6') - 2(\Gamma_0' + \Gamma_6') + 2\mathrm{i}\omega\mathcal{T}'\right).
\end{aligned} \tag{30}$$

By means of the algebra (30) and the relation (10) the spectrum of the 5D $SU(2)$ Kepler problem can be derived, as done recently in [8]. However, the group $SO(1,2)$ generated by (28) cannot be extended up to $SO(6,2)$ as in the case of the '8D harmonic oscillator–5D $SU(2)$ MIC-Kepler problem'. The situation is quite similar to that taking place in lower-dimensional consideration: the '4D harmonic oscillator–3D MIC-Kepler problem' [1].

In conclusion we show how one can derive the $SO(6)$ 'hidden' symmetry [6, 10] using the notation of (19). Notice that

$$\begin{aligned}
\mathcal{Y}_\mu &= r_\mu\left(\pi^2 + \frac{\boldsymbol{K}^2}{R^2} - \frac{2\kappa}{R}\right) + \frac{2\kappa r_\mu}{R} + \{\pi_\lambda, \mathcal{J}_{\lambda\mu}\} \\
&= 2r_\mu\mathcal{H}_0 + \frac{2\kappa r_\mu}{R} + \{\pi_\lambda, \mathcal{J}_{\lambda\mu}\}.
\end{aligned} \tag{31}$$

Then, one can define the operators

$$\mathcal{D}_\mu = \frac{1}{2}\mathcal{Y}_\mu - r_\mu \mathcal{H}_0 = \frac{1}{2}\{\pi_\lambda, \mathcal{J}_{\lambda\mu}\} + \frac{\kappa r_\mu}{R}$$
$$= \frac{\mathcal{M}_\mu + \mathcal{A}_\mu}{2} + \frac{\mathcal{M}_\mu - \mathcal{A}_\mu}{2}(-2\mathcal{H}_0) \qquad (32)$$

which satisfy

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = \mathrm{i}\mathcal{J}_{\mu\nu}(-2\mathcal{H}_0) \qquad (33)$$

or for the fixed energy level $\mathcal{H}_0 = \mathcal{E}_0$ one can introduce

$$\mathcal{D}'_\mu = \frac{\mathcal{D}_\mu}{\sqrt{-2\mathcal{E}_0}} \qquad (34)$$

which fit

$$[\mathcal{D}'_\mu, \mathcal{D}'_\nu] = \mathrm{i}\mathcal{J}_{\mu\nu}. \qquad (35)$$

The operators (34) along with $\mathcal{J}_{\mu\nu}$ constitute the algebra $SO(6)$.

The operators (19) generating the energy spectrum may be useful in many particle problems with quadrupole interaction manifesting $SO(5)$ symmetry, for example in nuclear physics [11].

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